

STEADY FREE FALL OF ONE-DIMENSIONAL BODIES IN A HYPERVISCOUS FLUID AT LOW REYNOLDS NUMBER

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ABSTRACT. The paper is devoted to the study of the motion of one-dimensional rigid bodies during a free fall in a quasi-Newtonian hyperviscous fluid at low Reynolds number. We show the existence of a steady solution and furnish sufficient conditions on the geometry of the body in order to get purely translational motions. Such conditions are based on a generalized version of the so-called *Reciprocal Theorem* for fluids.

1. INTRODUCTION

The study of the free fall of slender bodies in liquids is an essential issue in many problems of practical interest, such as the design of composite materials or the analytical technique of separation of macromolecules by electrophoresis (see [3] for a very interesting and rich review on the subject). Typical experiments show that homogeneous bodies satisfying some symmetry conditions, when dropped in a quiescent viscous liquid, will eventually reach a steady state that is purely translational, having the symmetry axis forming an angle with respect to the gravity \mathbf{g} , called *tilt angle*, that depends on the material geometry of the body and on the physical properties of the liquid.

If the geometry of the body is such that one of the dimensions dramatically prevails on the other two, the assumption that the body is one-dimensional is a reasonable simplification which can give satisfactory results. However, a one-dimensional body is “too thin” to interact with a classical Newtonian incompressible fluid in 3D (it has null capacity, see [9]). Hence, we propose to study the problem of the free fall of a slender body in a regularized model for Newtonian fluids, introduced by Fried and Gurtin [2] in 2006, where higher-order derivatives are considered in the constitutive prescription of the Cauchy stress tensor.

The Navier–Stokes equation for incompressible fluids reads

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \rho(\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p - \mu \Delta \mathbf{u} = \rho \mathbf{b},$$

where p is the pressure field, \mathbf{u} is the divergence-free velocity field, $\rho > 0$ is the constant and homogeneous mass density, $\mu > 0$ is the dynamic viscosity and $\rho \mathbf{b}$ is a volumetric force density. The *hyperviscous regularization* consists of adding a term proportional to $\Delta \Delta \mathbf{u}$ to the equation. For this modified equation,

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \rho(\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p - \mu \Delta \mathbf{u} + \zeta \Delta \Delta \mathbf{u} = \rho \mathbf{b},$$

where $\zeta > 0$ is the *hyperviscosity*, the existence and uniqueness of regular solutions have been established. In a series of papers [2, 8, 6, 4] a purely mechanical explanation of the hyperviscous term has been proposed, and different contributions to ζ associated with dissipation functionals are introduced and analyzed. Here we assign to ζ a geometric role, by introducing the *effective thickness* $L > 0$ of the lower-dimensional objects, and setting $\zeta = \mu L^2$, so that the hyperviscous flow equation becomes

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \rho(\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p - \mu \Delta (\mathbf{u} - L^2 \Delta \mathbf{u}) = \rho \mathbf{b}.$$

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In the experiments on the free fall of rigid bodies in viscous fluids the Reynolds number is often very small, so that the inertia of the liquid can be neglected and one can linearize the flow equation [9, 1]. However, even after that approximation the problem does not become fully linear, since there remains a nonlinear coupling between the flow and the rigid body motion.

In the present paper we will study the steady fall of one-dimensional rigid bodies in a hyperviscous fluid at low Reynolds number. In Section 2 we give the mathematical formulation of the problem, and in Section 3 we study the forces acting on the body in the case of a hyperviscous fluid. In Section 4 we show the existence of steady solutions, and in Section 5 we prove the Reciprocal Theorem in the case of a linearized hyperviscous liquid surrounding a one-dimensional body, and study some properties of the so-called *resistance tensors*. Finally, Section 6 contains sufficient conditions on the geometry of a homogeneous body in order to get purely translational solutions.

2. FORMULATION OF THE FREE FALL PROBLEM

The free fall problem is characterized by the fact that the rigid body is immersed and dropped from rest in an otherwise quiescent fluid and gravity is the only external force acting on the system. We represent a rigid body as a connected, bounded, closed subset Σ of \mathbb{R}^3 which is a finite union of images of $[0, 1]$ through a C^1 -diffeomorphism. It is convenient to write the problem in a co-moving frame, with origin at the center of mass $\mathbf{c}(t)$ of Σ . Denoting by \mathbf{y} the position of a point in the original inertial frame, and by \mathbf{x} its position in the co-moving frame, we know that, at any time $t \geq 0$,

$$\mathbf{x} = \mathbf{Q}^\top(t)(\mathbf{y} - \mathbf{c}(t)),$$

where $\mathbf{Q}(t)$ is an orthogonal linear transformation for any $t \geq 0$, with $\mathbf{Q}(0) = \mathbf{1}$. If the velocity of the center of mass and the spin of the rigid body in the inertial frame are denoted by $\boldsymbol{\eta}(t)$ and $\boldsymbol{\Omega}(t)$, respectively, so that

$$\mathbf{v}(t) = \boldsymbol{\eta}(t) + \boldsymbol{\Omega}(t) \times (\bar{\mathbf{y}} - \mathbf{c}(t))$$

is the velocity, in that frame, of any point $\bar{\mathbf{y}}$ belonging to the rigid body, then their expression in the co-moving frame is given by

$$\boldsymbol{\xi}(t) := \mathbf{Q}^\top(t)\boldsymbol{\eta}(t) \quad \text{and} \quad \boldsymbol{\omega}(t) := \mathbf{Q}^\top(t)\boldsymbol{\Omega}(t),$$

respectively, and the rigid velocity field \mathbf{v} is transformed into

$$\mathbf{U}(t) := \boldsymbol{\xi}(t) + \boldsymbol{\omega}(t) \times \bar{\mathbf{x}},$$

where $\bar{\mathbf{x}}$ denotes the coordinates of $\bar{\mathbf{y}}$ in the co-moving frame.

The gravitational acceleration vector \mathbf{g} (constant in the inertial frame) is represented, in the co-moving frame, by $\mathbf{G}(t) := \mathbf{Q}^\top(t)\mathbf{g}$, which is easily seen to satisfy the ordinary differential equation

$$\frac{d\mathbf{G}}{dt} = \mathbf{G} \times \boldsymbol{\omega}. \quad (1)$$

As customary when studying flows past rigid bodies, the velocity field \mathbf{u} that we consider is the so called *disturbance field*, which is the difference between the actual flow and the flow at infinity, both seen in the co-moving frame. Since the flow at infinity is $-\mathbf{U}$ (that is minus the extension to all of the fluid of the motion of the immersed object), the representation of the fluid flow in the co-moving frame is given by $\mathbf{u} - \mathbf{U}$.

The continuity and flow equations for an incompressible (disturbance) velocity field $\mathbf{u}(\mathbf{x}, t)$ and pressure field $p(\mathbf{x}, t)$ defined on $(\mathbb{R}^3 \setminus \Sigma) \times [0, +\infty)$, become then

$$\operatorname{div} \mathbf{u} = 0, \quad (2)$$

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + [(\mathbf{u} - \mathbf{U}) \cdot \nabla] \mathbf{u} + \boldsymbol{\omega} \times \mathbf{u} \right) = \operatorname{div} \mathbf{T}(\mathbf{u}, p) + \rho \mathbf{G}, \quad (3)$$

where $\mathbf{T}(\mathbf{u}, p)$ denotes the Cauchy stress tensor. Notice that, thanks to its frame indifference properties, \mathbf{T} retains the same functional dependence on the velocity field seen in both the inertial frame and in the co-moving one.

The disturbance field \mathbf{u} satisfies also the decay condition

$$\lim_{|\mathbf{x}| \rightarrow \infty} \mathbf{u}(\mathbf{x}, t) = \mathbf{0}, \quad (4)$$

and the adherence to the rigid body, given by

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{U}(\mathbf{x}, t) \quad \text{on } \Sigma \times [0, +\infty). \quad (5)$$

To properly account for Archimedean forces we introduce the *effective mass* of the body as given by

$$m_e = m - m_c,$$

that is, the difference between the real mass of the object and the *complementary mass* m_c of a portion of fluid occupying the real volume of the object. Even if for a one-dimensional body the complementary mass m_c should vanish, in view of the interpretation of such bodies as representations for real three-dimensional objects we allow for any value $0 \leq m_c \leq m$, suggested by the physical properties of the interaction between the body and the fluid. Then the equations of motion for the rigid body in the co-moving frame read

$$m \frac{d\boldsymbol{\xi}}{dt} + m\boldsymbol{\omega} \times \boldsymbol{\xi} = m_e \mathbf{G} + \mathbf{f}(\mathbf{u}, p), \quad (6)$$

$$\mathbf{J} \frac{d\boldsymbol{\omega}}{dt} + \boldsymbol{\omega} \times (\mathbf{J}\boldsymbol{\omega}) = -m_c \mathbf{r} \times \mathbf{G} + \mathbf{t}(\mathbf{u}, p), \quad (7)$$

where \mathbf{J} is the inertia tensor of Σ , \mathbf{r} is the position of the centroid⁽¹⁾ in the co-moving frame, and \mathbf{f} and \mathbf{t} are the total hydrodynamic force and torque, respectively, exerted on the rigid body as a consequence of the fluid flow \mathbf{u} and pressure p . The proper definitions of \mathbf{f} and \mathbf{t} , as well as the physical properties of the fluid encoded in the Cauchy stress tensor \mathbf{T} , are discussed in Section 3.

The whole set of equations (1)–(7) represents the differential problem associated with the free fall of a rigid object Σ in an incompressible fluid. It is convenient to consider it in the non-dimensional form: by choosing suitable reference speed W and length d , we can switch to non-dimensional quantities according to

$$\mathbf{x} \rightarrow \frac{\mathbf{x}}{d}, \quad t \rightarrow \frac{t\mu}{\rho d^2}, \quad \mathbf{u} \rightarrow \frac{\mathbf{u}}{W}, \quad p \rightarrow \frac{pd}{\mu W}, \quad m \rightarrow \frac{m}{\rho d^3},$$

obtaining

$$\operatorname{div} \mathbf{u} = 0, \quad (8)$$

$$\frac{\partial \mathbf{u}}{\partial t} + Re \{[(\mathbf{u} - \mathbf{U}) \cdot \nabla] \mathbf{u} + \boldsymbol{\omega} \times \mathbf{u}\} = \operatorname{div} \mathbf{T}(\mathbf{u}, p) + \mathbf{G}, \quad (9)$$

$$\lim_{|\mathbf{x}| \rightarrow \infty} \mathbf{u}(\mathbf{x}, t) = \mathbf{0}, \quad (10)$$

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{U}(\mathbf{x}, t) \quad \text{on } \Sigma \times [0, +\infty), \quad (11)$$

$$m \frac{d\boldsymbol{\xi}}{dt} + Re(m\boldsymbol{\omega} \times \boldsymbol{\xi}) = m_e \mathbf{G} + \mathbf{f}(\mathbf{u}, p), \quad (12)$$

$$\mathbf{J} \frac{d\boldsymbol{\omega}}{dt} + Re[\boldsymbol{\omega} \times (\mathbf{J}\boldsymbol{\omega})] = -m_c \mathbf{r} \times \mathbf{G} + \mathbf{t}(\mathbf{u}, p), \quad (13)$$

$$\frac{d\mathbf{G}}{dt} = Re(\mathbf{G} \times \boldsymbol{\omega}), \quad (14)$$

with initial conditions

$$\mathbf{u}(\mathbf{x}, 0) = \boldsymbol{\xi}(0) = \boldsymbol{\omega}(0) = \mathbf{0}, \quad \mathbf{G}(0) = \mathbf{g}, \quad (15)$$

where $Re = \rho W d / \mu$ is the Reynolds number and every quantity has to be understood as non-dimensional. Moreover, a suitable choice of the reference quantities ensures that $|\mathbf{G}(t)| = 1$ for any $t \geq 0$.

⁽¹⁾Notice that the centroid of a rigid body coincides with its center of mass when the body has a uniform mass density; in the latter case $\mathbf{r} = \mathbf{0}$.

The low-Reynolds-number approximation of the differential problem, which is also a linearization of the equation for the flow, is obtained by neglecting the terms proportional to Re in equations (9), (12), and (13), whereas equation (14) remains unchanged, since it represents a geometric constraint which holds for any non-vanishing value of Re . Moreover, the steady version of the problem is achieved by assuming that all the quantities do not depend on time, hence neglecting all the time derivatives. In that case, keeping into account also the initial conditions (15), equation (9) becomes

$$\operatorname{div} \mathbb{T}(\mathbf{u}, p) + \mathbf{g} = 0, \quad (16)$$

and equation (11) writes

$$\mathbf{u}(\mathbf{x}) = \mathbf{U}(\mathbf{x}) = \boldsymbol{\xi} + \boldsymbol{\omega} \times \mathbf{x} \quad \text{on } \Sigma.$$

Finally, equations (12)–(14) become

$$m_e \mathbf{g} + \mathbf{f}(\mathbf{u}, p) = 0, \quad -m_c \mathbf{r} \times \mathbf{g} + \mathbf{t}(\mathbf{u}, p) = 0, \quad \mathbf{g} \times \boldsymbol{\omega} = \mathbf{0},$$

respectively.

3. THE VISCOUS FORCE ACTING ON A SLENDER BODY

Given $r > 0$, we introduce the (closed) r -neighborhood of the slender body Σ by setting

$$V_r(\Sigma) := \left\{ \mathbf{x} \in \mathbb{R}^3 : \inf_{\mathbf{c} \in \Sigma} |\mathbf{x} - \mathbf{c}| \leq r \right\}.$$

Then we define the total hydrodynamic force, due to the fluid velocity and pressure field (\mathbf{u}, p) , acting on Σ as

$$\mathbf{f}(\mathbf{u}, p) := \lim_{r \rightarrow 0} \int_{\partial V_r(\Sigma)} \mathbb{T}(\mathbf{u}, p) \mathbf{n}, \quad (17)$$

where \mathbf{n} denotes the unit outer normal to $\partial V_r(\Sigma)$. Notice that, thanks to the regularity of Σ , the r -neighborhood $V_r(\Sigma)$ has a Lipschitz boundary for any r sufficiently small.

Proposition 3.1. *The limit in (17) is well-defined.*

Proof. We consider a ball B_R centered at the origin and with radius R , which contains $V_r(\Sigma)$ for some $r > 0$. According to equation (16), the term $\operatorname{div} \mathbb{T}$ balances the gravity, so that it is represented by a measure whose singular part is concentrated on Σ (we can see this by noting that the mass density per unit volume must diverge on Σ to give a non-zero weight to a body with vanishing volume). Denote by λ_{ac} and λ_s the absolutely continuous and singular parts of the measure $\operatorname{div} \mathbb{T}$, respectively. It follows that the support of λ_s is contained in Σ and that $\operatorname{div} \mathbb{T} = \lambda_{ac}$ in $\mathbb{R}^3 \setminus \Sigma$. Then we have, by applying Lebesgue's theorem,

$$\lim_{r \rightarrow 0} \int_{B_R \setminus V_r(\Sigma)} \operatorname{div} \mathbb{T} = \lim_{r \rightarrow 0} \int_{B_R \setminus V_r(\Sigma)} \lambda_{ac} = \int_{B_R \setminus \Sigma} \lambda_{ac} = \int_{B_R \setminus \Sigma} \operatorname{div} \mathbb{T}, \quad (18)$$

since $\lambda_{ac} \in L^1(\mathbb{R}^3; \mathbb{R}^3)$. Then, by the Divergence theorem,

$$\mathbf{f} = \lim_{r \rightarrow 0} \int_{\partial V_r(\Sigma)} \mathbb{T} \mathbf{n} = \int_{\partial B_R} \mathbb{T} \mathbf{n} - \int_{B_R \setminus \Sigma} \operatorname{div} \mathbb{T}, \quad (19)$$

where \mathbf{n} is always the outer normal. Since the right-hand side of (19) is independent of r , the left-hand side is well defined. \square

In a similar fashion, we define the total hydrodynamic torque acting on Σ , due to the fluid velocity and pressure field (\mathbf{u}, p) , as

$$\mathbf{t}(\mathbf{u}, p) := \lim_{r \rightarrow 0} \int_{\partial V_r(\Sigma)} \mathbf{x} \times \mathbb{T}(\mathbf{u}, p) \mathbf{n}.$$

It is important to stress the fact that, if λ_s were absent, *i.e.* if $\operatorname{div} \mathbb{T}$ were an L^1 -function, then the integral over $B_R \setminus \Sigma$ of $\operatorname{div} \mathbb{T}$ in (19) would be equal to its integral over all of B_R and \mathbf{f} and \mathbf{t} would simply vanish.

The constitutive theory for non-simple fluids leading to a hyperviscous flow equation has been developed in [2, 8, 6]. It offers a number of possible choices for the terms to be included in \mathbb{T} , in addition to those of Newtonian fluids. Here we make the following, somewhat minimal, choice:

$$\mathbb{T}(\mathbf{u}, p) := -p\mathbf{1} + (\nabla \mathbf{u} + \nabla \mathbf{u}^\top - \ell^2 \nabla \Delta \mathbf{u}). \quad (20)$$

In this way we obtain a fluid which is quasi-Newtonian, while being able to adhere to lower-dimensional objects. The only new parameter ℓ is given by L/d , hence it is non-dimensional and strictly positive; as explained in [5], it can be assigned the geometric meaning of an effective thickness of the slender body Σ . It is straightforward to check that \mathbb{T} , as defined in (20), enjoys the standard symmetry and frame indifference properties and satisfies a dissipation inequality. In particular, one has

$$\operatorname{div} \mathbb{T} = -\nabla p + \Delta \mathbf{u} - \ell^2 \Delta \Delta \mathbf{u}.$$

We summarize the problem of the steady free fall of a one-dimensional body Σ at low Reynolds number, as the following: find $(\mathbf{u}, p) \in H^2(\mathbb{R}^3; \mathbb{R}^3) \times H^{-1}(\mathbb{R}^3; \mathbb{R})$ and $\boldsymbol{\xi}, \boldsymbol{\omega}, \mathbf{g} \in \mathbb{R}^3$ with $|\mathbf{g}| = 1$, such that

$$\operatorname{div} \mathbf{u} = 0, \quad (21)$$

$$\nabla p - \Delta \mathbf{u} + \ell^2 \Delta \Delta \mathbf{u} = \mathbf{g} \quad \text{on } \mathbb{R}^3 \setminus \Sigma, \quad (22)$$

$$\mathbf{u}(\mathbf{x}) = \boldsymbol{\xi} + \boldsymbol{\omega} \times \mathbf{x} \quad \text{on } \Sigma, \quad (23)$$

$$m_e \mathbf{g} = -\mathbf{f}, \quad (24)$$

$$m_e \mathbf{r} \times \mathbf{g} = \mathbf{t}, \quad (25)$$

$$\mathbf{g} \times \boldsymbol{\omega} = \mathbf{0}. \quad (26)$$

The set H^k is the usual Sobolev space of functions with square-integrable weak derivatives up to the order k , and the set H^{-k} is the topological dual of H^k ; the strong decay condition (4) is encoded as an integrability condition in the assumption $\mathbf{u} \in H^2(\mathbb{R}^3; \mathbb{R}^3)$. Notice that, although equation (22) is linear, the full problem is nonlinear.

4. STEADY FREE FALL AT LOW REYNOLDS NUMBER

In this section we prove the existence of a solution for the the differential problem (21)–(26). We begin by introducing some auxiliary problems, which are well-posed by virtue of the following result.

Lemma 4.1. *Given $\boldsymbol{\xi}, \boldsymbol{\omega} \in \mathbb{R}^3$, there exists a unique solution $(\mathbf{h}, p) \in H^2(\mathbb{R}^3; \mathbb{R}^3) \times H^{-1}(\mathbb{R}^3; \mathbb{R})$ of the problem*

$$\begin{cases} \operatorname{div} \mathbf{h} = 0 & \text{in } \mathbb{R}^3, \\ \nabla p - \Delta \mathbf{h} + \ell^2 \Delta \Delta \mathbf{h} = 0 & \text{in } \mathbb{R}^3, \\ \mathbf{h} = \boldsymbol{\xi} + \boldsymbol{\omega} \times \mathbf{x} & \text{on } \Sigma. \end{cases} \quad (27)$$

Moreover, such a solution is of class C^∞ on $\mathbb{R}^3 \setminus \Sigma$.

Proof. Since $H^2(\mathbb{R}^3; \mathbb{R}^3)$ embeds in a space of Hölder-continuous functions, the subset

$$H := \{\mathbf{v} \in H^2(\mathbb{R}^3; \mathbb{R}^3) : \operatorname{div} \mathbf{v} = 0 \text{ and } \mathbf{v} = \boldsymbol{\xi} + \boldsymbol{\omega} \times \mathbf{x} \text{ on } \Sigma\}$$

is well-defined, closed and convex. The velocity field \mathbf{h} can be found by minimizing on H the functional

$$\mathcal{F}(\mathbf{v}) := \frac{1}{2} \int_{\mathbb{R}^3} (2|\operatorname{Sym} \nabla \mathbf{v}|^2 + \ell^2 |\Delta \mathbf{v}|^2),$$

where $\operatorname{Sym} \nabla \mathbf{v} := (\nabla \mathbf{v} + \nabla \mathbf{v}^\top)/2$. Being \mathcal{F} a strictly convex functional, \mathbf{h} is unique. Then, the pressure field p can be recovered as the Lagrange multiplier of the divergence-free constraint.

The same technique used to study the regularity of the solution for the classical Stokes equation can be applied here to prove that the solution is of class C^∞ on $\mathbb{R}^3 \setminus \Sigma$. \square

Consider now the solutions $(\mathbf{h}^{(i)}, p^{(i)})$ and $(\mathbf{H}^{(i)}, P^{(i)})$ ($i = 1, 2, 3$) of

$$\begin{cases} \operatorname{div} \mathbf{h}^{(i)} = 0 & \text{in } \mathbb{R}^3, \\ \nabla p^{(i)} - \Delta \mathbf{h}^{(i)} + \ell^2 \Delta \Delta \mathbf{h}^{(i)} = 0 & \text{in } \mathbb{R}^3, \\ \mathbf{h}^{(i)} = \mathbf{e}_i & \text{on } \Sigma, \end{cases} \quad (28)$$

and

$$\begin{cases} \operatorname{div} \mathbf{H}^{(i)} = 0 & \text{in } \mathbb{R}^3, \\ \nabla P^{(i)} - \Delta \mathbf{H}^{(i)} + \ell^2 \Delta \Delta \mathbf{H}^{(i)} = 0 & \text{in } \mathbb{R}^3, \\ \mathbf{H}^{(i)} = \mathbf{e}_i \times \mathbf{x} & \text{on } \Sigma. \end{cases} \quad (29)$$

We will show that the combinations

$$\mathbf{u} = \sum_{i=1}^3 [\xi_i \mathbf{h}^{(i)} + \omega_i \mathbf{H}^{(i)}], \quad p = \sum_{i=1}^3 [\xi_i p^{(i)} + \omega_i P^{(i)}] + \mathbf{g} \cdot \mathbf{x}, \quad (30)$$

for a suitable choice of the vectors $\boldsymbol{\xi}$ and $\boldsymbol{\omega}$, solve the steady free fall problem. First we need to introduce four matrices, which will play an even more important role in Sections 5 and 6.

Definition 4.2. The matrices \mathbf{K} , \mathbf{S} , \mathbf{C} , and \mathbf{B} are defined in Cartesian components by

$$\mathbf{K}_{ji} := - \lim_{r \rightarrow 0} \int_{\partial V_r(\Sigma)} \mathbf{T}(\mathbf{h}^{(i)}, p^{(i)}) \mathbf{n} \cdot \mathbf{e}_j, \quad (31)$$

$$\mathbf{S}_{ji} := - \lim_{r \rightarrow 0} \int_{\partial V_r(\Sigma)} \mathbf{T}(\mathbf{H}^{(j)}, P^{(j)}) \mathbf{n} \cdot \mathbf{e}_i, \quad (32)$$

$$\mathbf{C}_{ji} := - \lim_{r \rightarrow 0} \int_{\partial V_r(\Sigma)} \mathbf{x} \times \mathbf{T}(\mathbf{h}^{(j)}, p^{(j)}) \mathbf{n} \cdot \mathbf{e}_i, \quad (33)$$

$$\mathbf{B}_{ji} := - \lim_{r \rightarrow 0} \int_{\partial V_r(\Sigma)} \mathbf{x} \times \mathbf{T}(\mathbf{H}^{(i)}, P^{(i)}) \mathbf{n} \cdot \mathbf{e}_j, \quad (34)$$

where \mathbf{n} is the outer normal to $V_r(\Sigma)$. Following [1], they are called *resistance tensors*, and in particular \mathbf{K} is the *translation tensor*, \mathbf{B} the *rotation tensor*, and \mathbf{S} and \mathbf{C} the *coupling tensors*. Moreover, we denote by \mathbf{A} the 6×6 matrix

$$\mathbf{A} := \begin{pmatrix} \mathbf{K} & \mathbf{S} \\ \mathbf{C} & \mathbf{B} \end{pmatrix}.$$

We postpone to Theorems 5.3 and 5.4, in the next section, the proof of a fundamental property:
the matrices \mathbf{K} , \mathbf{B} and \mathbf{A} are symmetric and positive definite.

Although an energetic argument of Brenner [7, Section 5–2] is usually adopted in this case, we will prefer to give a direct proof.

Now we can prove the main theorem of the section.

Theorem 4.3 (Existence Theorem). *The differential problem (21)–(26) admits a solution $(\mathbf{u}, p, \boldsymbol{\xi}, \boldsymbol{\omega}, \mathbf{g})$.*

Proof. It is straightforward to check that the fields \mathbf{u} and p defined by (30) satisfy equations (21), (22), and (23). Equation (26) implies that $\boldsymbol{\omega} = \lambda \mathbf{g}$ for some $\lambda \in \mathbb{R}$, and equations (24) and (25) reduce to the following algebraic system in the six scalar unknowns $\boldsymbol{\xi}$, λ , and \mathbf{g} (recall that $|\mathbf{g}| = 1$):

$$\begin{cases} \mathbf{K} \boldsymbol{\xi} + \lambda \mathbf{S} \mathbf{g} = m_e \mathbf{g} \\ \mathbf{C} \boldsymbol{\xi} + \lambda \mathbf{B} \mathbf{g} = -m_e \mathbf{r} \times \mathbf{g}. \end{cases} \quad (35)$$

It is now clear that the steady free fall problem admits a solution if and only if (35) admits a solution, and the latter fact is related to the properties of the matrix

$$\mathbf{A} = \begin{pmatrix} \mathbf{K} & \mathbf{S} \\ \mathbf{C} & \mathbf{B} \end{pmatrix}.$$

Since \mathbf{K} is non singular, the first equation of (35) becomes

$$\boldsymbol{\xi} = \mathbf{K}^{-1}(m_e \mathbf{g} - \lambda \mathbf{S} \mathbf{g})$$

and one can eliminate $\boldsymbol{\xi}$ in the second equation of (35). Since \mathbf{A} is non-singular, the linear transformation

$$\mathbf{F}\mathbf{g} := (\mathbf{C}\mathbf{K}^{-1}\mathbf{S} - \mathbf{B})^{-1}(m_e\mathbf{C}\mathbf{K}^{-1}\mathbf{g} + m_e\mathbf{r} \times \mathbf{g})$$

is well-defined and non-singular, and we can write (35) as

$$\begin{cases} \boldsymbol{\xi} = \mathbf{K}^{-1}(m_e\mathbf{g} + \lambda\mathbf{S}\mathbf{g}) \\ \mathbf{F}\mathbf{g} = \lambda\mathbf{g}. \end{cases} \quad (36)$$

Being \mathbf{F} a 3×3 real matrix, it has at least one real eigenvalue. Such an eigenvalue λ , the associated unit eigenvector \mathbf{g} and $\boldsymbol{\xi}$ calculated as in the first equation of (36), together with the fields \mathbf{u} and p introduced in (30), furnish a solution for equations (21)–(26). \square

5. AN ANALYSIS OF THE RESISTANCE TENSORS

In (31)–(34) we introduced the four resistance tensors \mathbf{K} , \mathbf{S} , \mathbf{C} and \mathbf{B} . In view of the conditions on Σ assumed in the auxiliary problems (28)–(29), we can give an equivalent characterization.

Proposition 5.1. *The resistance tensors are such that*

$$\begin{aligned} \mathbf{K}_{ji} &= -\lim_{r \rightarrow 0} \int_{\partial V_r(\Sigma)} \mathbf{T}(\mathbf{h}^{(i)}, p^{(i)}) \mathbf{n} \cdot \mathbf{h}^{(j)}, \\ \mathbf{S}_{ji} &= -\lim_{r \rightarrow 0} \int_{\partial V_r(\Sigma)} \mathbf{T}(\mathbf{H}^{(j)}, P^{(j)}) \mathbf{n} \cdot \mathbf{h}^{(i)}, \\ \mathbf{C}_{ji} &= -\lim_{r \rightarrow 0} \int_{\partial V_r(\Sigma)} \mathbf{T}(\mathbf{h}^{(j)}, p^{(j)}) \mathbf{n} \cdot \mathbf{H}^{(i)}, \\ \mathbf{B}_{ji} &= -\lim_{r \rightarrow 0} \int_{\partial V_r(\Sigma)} \mathbf{T}(\mathbf{H}^{(i)}, P^{(i)}) \mathbf{n} \cdot \mathbf{H}^{(j)}. \end{aligned}$$

Proof. Since $\mathbf{h}^{(i)}$ is smooth and $\mathbf{h}^{(i)} = \mathbf{e}_i$ on Σ , one has, as $r \rightarrow 0$,

$$\|\mathbf{h}^{(i)} - \mathbf{e}_i\|_{\infty, \partial V_r(\Sigma)} := \sup_{\mathbf{x} \in \partial V_r(\Sigma)} |\mathbf{h}^{(i)}(\mathbf{x}) - \mathbf{e}_i| \rightarrow 0.$$

Hence, considering for instance the translation tensor \mathbf{K} , it follows that

$$\begin{aligned} & \left| \int_{\partial V_r(\Sigma)} \mathbf{T}(\mathbf{h}^{(i)}, p^{(i)}) \mathbf{n} \cdot \mathbf{h}^{(j)} - \int_{\partial V_r(\Sigma)} \mathbf{T}(\mathbf{h}^{(i)}, p^{(i)}) \mathbf{n} \cdot \mathbf{e}_j \right| \\ & \leq \|\mathbf{h}^{(j)} - \mathbf{e}_j\|_{\infty, \partial V_r(\Sigma)} \int_{\partial V_r(\Sigma)} |\mathbf{T}(\mathbf{h}^{(i)}, p^{(i)}) \mathbf{n}| \rightarrow 0 \end{aligned}$$

as $r \rightarrow 0$. The proof of the other three formulae is similar. \square

In the remainder of the section we will prove that \mathbf{K} and \mathbf{B} are symmetric and that $\mathbf{C}^\top = \mathbf{S}$. We first need a fundamental property of steady incompressible flows at low Reynolds number, the so-called *Reciprocal Theorem* (see [7, Sec. 3-5]), which, roughly speaking, states a reciprocity property between two solutions of the same equation, independently of the boundary conditions. The validity of the theorem, which is quite trivial for ordinary fluids, is not so obvious in the present case of hyperviscous fluids, since the lack of further boundary conditions and the higher order of the differential operator can break such a reciprocity. However, the theorem can be recovered for the particular case of one-dimensional bodies.

Theorem 5.2 (Reciprocal Theorem). *Given two solutions (\mathbf{u}_1, p_1) and (\mathbf{u}_2, p_2) in the space $H^2(\mathbb{R}^3; \mathbb{R}^3) \times H^{-1}(\mathbb{R}^3; \mathbb{R})$ of the equations*

$$\operatorname{div} \mathbf{u} = 0 \quad \text{in } \mathbb{R}^3, \quad \operatorname{div} \mathbf{T}(\mathbf{u}, p) = 0 \quad \text{in } \mathbb{R}^3 \setminus \Sigma, \quad (37)$$

where $\mathbf{T}(\mathbf{u}, p)$ is defined as in (20), we have

$$\lim_{r \rightarrow 0} \int_{\partial V_r(\Sigma)} \mathbf{T}(\mathbf{u}_1, p_1) \mathbf{n} \cdot \mathbf{u}_2 = \lim_{r \rightarrow 0} \int_{\partial V_r(\Sigma)} \mathbf{T}(\mathbf{u}_2, p_2) \mathbf{n} \cdot \mathbf{u}_1.$$

Proof. Consider a large ball B_R containing $V_r(\Sigma)$ and apply Gauss-Green formula to the domain $B_R \setminus V_r(\Sigma)$: then

$$\int_{\partial V_r(\Sigma)} \mathbf{T}(\mathbf{u}_1, p_1) \mathbf{n} \cdot \mathbf{u}_2 = - \int_{B_R \setminus V_r(\Sigma)} \mathbf{T}(\mathbf{u}_1, p_1) \cdot \nabla \mathbf{u}_2 - \int_{\partial B_R} \mathbf{T}(\mathbf{u}_1, p_1) \mathbf{n} \cdot \mathbf{u}_2,$$

where the normal in the left-hand side is exterior to $V_r(\Sigma)$ and we kept into account that $\operatorname{div} \mathbf{T}(\mathbf{u}_2, p_2) = 0$ on $\mathbb{R}^3 \setminus \Sigma$. The last surface integral on ∂B_R vanishes as $R \rightarrow +\infty$, since any solution to the hyperviscous Stokes' problem decays as $1/|\mathbf{x}|$ (see Appendix A), hence \mathbf{T} decays as $1/|\mathbf{x}|^2$.

Now take the first term of the right-hand side and use the constitutive prescription (20):

$$\begin{aligned} - \int_{B_R \setminus V_r(\Sigma)} \mathbf{T}(\mathbf{u}_1, p_1) \cdot \nabla \mathbf{u}_2 &= - \int_{B_R \setminus V_r(\Sigma)} (\nabla \mathbf{u}_1 + \nabla \mathbf{u}_1^\top) \cdot \nabla \mathbf{u}_2 + \ell^2 \int_{B_R \setminus V_r(\Sigma)} \nabla \Delta \mathbf{u}_1 \cdot \nabla \mathbf{u}_2 \\ &= - \int_{B_R \setminus V_r(\Sigma)} (\nabla \mathbf{u}_2 + \nabla \mathbf{u}_2^\top) \cdot \nabla \mathbf{u}_1 + \ell^2 \int_{B_R \setminus V_r(\Sigma)} \nabla \Delta \mathbf{u}_1 \cdot \nabla \mathbf{u}_2. \end{aligned}$$

Consider the last term; since the gradient and Laplace operators commute, by Green's second identity it follows that

$$\begin{aligned} \int_{B_R \setminus V_r(\Sigma)} \nabla \Delta \mathbf{u}_1 \cdot \nabla \mathbf{u}_2 &= \int_{B_R \setminus V_r(\Sigma)} \nabla \mathbf{u}_1 \cdot \nabla \Delta \mathbf{u}_2 \\ &\quad - \int_{\partial V_r(\Sigma)} [\nabla \mathbf{u}_2 \cdot (\nabla \nabla \mathbf{u}_1) \mathbf{n} - \nabla \mathbf{u}_1 \cdot (\nabla \nabla \mathbf{u}_2) \mathbf{n}] \\ &\quad + \int_{\partial B_R} [\nabla \mathbf{u}_2 \cdot (\nabla \nabla \mathbf{u}_1) \mathbf{n} - \nabla \mathbf{u}_1 \cdot (\nabla \nabla \mathbf{u}_2) \mathbf{n}]. \end{aligned}$$

Now we claim that the surface integrals vanish as $r \rightarrow 0$ and $R \rightarrow +\infty$. Indeed, take for instance the term

$$\int_{\partial V_r(\Sigma)} \nabla \mathbf{u}_2 \cdot (\nabla \nabla \mathbf{u}_1) \mathbf{n}$$

and apply again Gauss-Green formula inside $V_r(\Sigma)$. Then

$$\int_{\partial V_r(\Sigma)} \nabla \mathbf{u}_2 \cdot (\nabla \nabla \mathbf{u}_1) \mathbf{n} = \int_{V_r(\Sigma)} [\nabla \nabla \mathbf{u}_1 \cdot \nabla \nabla \mathbf{u}_2 + \nabla \mathbf{u}_2 \cdot \Delta \nabla \mathbf{u}_1].$$

Since $\mathbf{u}_1, \mathbf{u}_2 \in H^2(\mathbb{R}^3; \mathbb{R}^3)$, then $\nabla \nabla \mathbf{u}_1 \cdot \nabla \nabla \mathbf{u}_2 \in L^1(\mathbb{R}^3)$ and

$$\lim_{r \rightarrow 0} \int_{V_r(\Sigma)} \nabla \nabla \mathbf{u}_1 \cdot \nabla \nabla \mathbf{u}_2 = 0.$$

In a similar way, since $\nabla \mathbf{u}_2 \in H^1(\mathbb{R}^3; \operatorname{Mat}_{3 \times 3})$ and $\Delta \nabla \mathbf{u}_1 \in H^{-1}(\mathbb{R}^3; \operatorname{Mat}_{3 \times 3})$, then $\nabla \mathbf{u}_2 \cdot \Delta \nabla \mathbf{u}_1 \in L^1(\mathbb{R}^3)$ and

$$\lim_{r \rightarrow 0} \int_{V_r(\Sigma)} \nabla \mathbf{u}_2 \cdot \Delta \nabla \mathbf{u}_1 = 0.$$

Regarding the surface integral on ∂B_R , it vanishes as $R \rightarrow +\infty$ since $\mathbf{u}_1, \mathbf{u}_2$ decay as $1/|\mathbf{x}|$ (see Appendix A). \square

By combining Proposition 5.1 with Theorem 5.2 one can immediately prove the main result of the section:

Theorem 5.3. *The resistance tensors are such that*

$$\mathbf{K}^\top = \mathbf{K}, \quad \mathbf{B}^\top = \mathbf{B}, \quad \mathbf{C}^\top = \mathbf{S}.$$

In particular, the matrix \mathbf{A} is symmetric.

Now we are in a position to give a proof of the positive definiteness of the tensors \mathbf{K} , \mathbf{B} and \mathbf{A} .

Theorem 5.4. *The 6×6 matrix \mathbf{A} is positive definite. As a consequence, also \mathbf{K} and \mathbf{B} are positive definite.*

Proof. For $\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3)$ and $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$ set

$$\mathbf{u} = \sum_{i=1}^3 [\xi_i \mathbf{h}^{(i)} + \omega_i \mathbf{H}^{(i)}], \quad \mathbf{p} = \sum_{i=1}^3 [\xi_i \mathbf{p}^{(i)} + \omega_i \mathbf{P}^{(i)}],$$

where $(\mathbf{h}^{(i)}, \mathbf{p}^{(i)})$ and $(\mathbf{H}^{(i)}, \mathbf{P}^{(i)})$ are the solutions of the auxiliary problems (28)–(29). Using Proposition 5.1, the Reciprocal Theorem and the linearity of \mathbf{T} one can check that

$$\begin{pmatrix} \boldsymbol{\xi} \\ \boldsymbol{\omega} \end{pmatrix} \cdot \mathbf{A} \begin{pmatrix} \boldsymbol{\xi} \\ \boldsymbol{\omega} \end{pmatrix} = \boldsymbol{\xi} \cdot \mathbf{K} \boldsymbol{\xi} + 2\boldsymbol{\omega} \cdot \mathbf{C} \boldsymbol{\xi} + \boldsymbol{\omega} \cdot \mathbf{B} \boldsymbol{\omega} = - \lim_{r \rightarrow 0} \int_{\partial V_r(\Sigma)} \mathbf{T}(\mathbf{u}, \mathbf{p}) \mathbf{n} \cdot \mathbf{u}.$$

Now argue as in the proof of the Reciprocal Theorem. Take a large ball B_R containing $V_r(\Sigma)$ and apply Gauss-Green formula to the domain $B_R \setminus V_r(\Sigma)$ to obtain

$$- \int_{\partial V_r(\Sigma)} \mathbf{T}(\mathbf{u}, \mathbf{p}) \mathbf{n} \cdot \mathbf{u} = \int_{B_R \setminus V_r(\Sigma)} \mathbf{T}(\mathbf{u}, \mathbf{p}) \cdot \nabla \mathbf{u} + \int_{\partial B_R} \mathbf{T}(\mathbf{u}, \mathbf{p}) \mathbf{n} \cdot \mathbf{u},$$

where the normal in the left-hand side is exterior to $V_r(\Sigma)$ and we kept into account that $\operatorname{div} \mathbf{T}(\mathbf{u}, \mathbf{p}) = 0$ on $\mathbb{R}^3 \setminus \Sigma$. The last surface integral on ∂B_R vanishes as $R \rightarrow +\infty$ since the solution decays as $1/|\mathbf{x}|$, hence \mathbf{T} decays as $1/|\mathbf{x}|^2$.

Now consider the first term of the right-hand side and use the constitutive prescription (20):

$$\begin{aligned} \int_{B_R \setminus V_r(\Sigma)} \mathbf{T}(\mathbf{u}, \mathbf{p}) \cdot \nabla \mathbf{u} &= \int_{B_R \setminus V_r(\Sigma)} (\nabla \mathbf{u} + \nabla \mathbf{u}^\top) \cdot \nabla \mathbf{u} - \ell^2 \int_{B_R \setminus V_r(\Sigma)} \nabla \Delta \mathbf{u} \cdot \nabla \mathbf{u} \\ &= \int_{B_R \setminus V_r(\Sigma)} |\nabla \mathbf{u}|^2 + \int_{B_R \setminus V_r(\Sigma)} \nabla \mathbf{u}^\top \cdot \nabla \mathbf{u} - \ell^2 \int_{B_R \setminus V_r(\Sigma)} \nabla \Delta \mathbf{u} \cdot \nabla \mathbf{u}. \end{aligned} \quad (38)$$

We deal with the second integral, taking into account that $\operatorname{div} \mathbf{u} = 0$:

$$\int_{B_R \setminus V_r(\Sigma)} \nabla \mathbf{u}^\top \cdot \nabla \mathbf{u} = - \int_{\partial V_r(\Sigma)} \mathbf{u} \cdot (\nabla \mathbf{u}^\top \mathbf{n}) + \int_{\partial B_R} \mathbf{u} \cdot (\nabla \mathbf{u}^\top \mathbf{n}).$$

The last integral on ∂B_R vanishes as $R \rightarrow \infty$ for the usual asymptotic behavior at infinity. By applying the Gauss-Green theorem on the other integral it follows that

$$- \int_{\partial V_r(\Sigma)} \mathbf{u} \cdot (\nabla \mathbf{u}^\top \mathbf{n}) = - \int_{V_r(\Sigma)} \nabla \mathbf{u} \cdot \nabla \mathbf{u}^\top - \int_{V_r(\Sigma)} \mathbf{u} \cdot \operatorname{div}(\nabla \mathbf{u}^\top)$$

and both terms vanish as $r \rightarrow 0$ since the first integrand is in L^1 and in the last integral one has $\operatorname{div} \mathbf{u} = 0$.

Now we deal with the last integral of (38). Since the gradient and Laplace operators commute, we have

$$\begin{aligned} -\ell^2 \int_{B_R \setminus V_r(\Sigma)} \nabla \Delta \mathbf{u} \cdot \nabla \mathbf{u} &= \ell^2 \int_{B_R \setminus V_r(\Sigma)} |\nabla \nabla \mathbf{u}|^2 \\ &\quad + \ell^2 \int_{\partial V_r(\Sigma)} \nabla \mathbf{u} \cdot (\nabla \nabla \mathbf{u}) \mathbf{n} - \ell^2 \int_{\partial B_R} \nabla \mathbf{u} \cdot (\nabla \nabla \mathbf{u}) \mathbf{n}. \end{aligned}$$

Following the last part of the proof of the Reciprocal Theorem, it can be proved that the surface integrals vanish as $r \rightarrow 0$ and $R \rightarrow +\infty$. Summarizing,

$$\begin{pmatrix} \boldsymbol{\xi} \\ \boldsymbol{\omega} \end{pmatrix} \cdot \mathbf{A} \begin{pmatrix} \boldsymbol{\xi} \\ \boldsymbol{\omega} \end{pmatrix} = \int_{B_R \setminus V_r(\Sigma)} |\nabla \mathbf{u}|^2 + \ell^2 \int_{B_R \setminus V_r(\Sigma)} |\nabla \nabla \mathbf{u}|^2,$$

hence \mathbf{A} is (strictly) positive definite. \square

6. TRANSLATIONAL SOLUTIONS FOR BODIES WITH SYMMETRIES

The free fall of a one-dimensional body in a hyperviscous fluid at low Reynolds number is characterized by 21 independent coefficients: 12 coefficients for the tensors \mathbf{K} and \mathbf{B} and 9 coefficients for the coupling tensor \mathbf{C} . However, material symmetries of the body can significantly reduce such a number. Moreover, the symmetries induce some restrictions on the form of the resistance tensors. We are specifically interested in symmetries which induces purely translational motions of the body (that is, with $\boldsymbol{\omega} = 0$). We will now study some particular symmetries.

If the body is invariant under a change of frame given by an orthogonal matrix \mathbf{Q} , then also the solutions $\mathbf{h}^{(i)}$ of the auxiliary problems (28) do not change; on the contrary, the solutions $\mathbf{H}^{(i)}$ of (29) undergo a sign change if $\det \mathbf{Q} = -1$, due to the presence of the vector product in the boundary condition. Hence one can prove that

$$\mathbf{K} = \mathbf{Q}^\top \mathbf{K} \mathbf{Q}, \quad \mathbf{B} = \mathbf{Q}^\top \mathbf{B} \mathbf{Q}, \quad \mathbf{C} = (\det \mathbf{Q}) \mathbf{Q}^\top \mathbf{C} \mathbf{Q}. \quad (39)$$

6.1. Bodies with a plane of symmetry. We say that the body Σ has x_2x_3 as a *plane of material symmetry*, if the density function ρ of the body satisfies

$$\rho(-x_1, x_2, x_3) = \rho(x_1, x_2, x_3) \quad \text{for every } (x_1, x_2, x_3) \in \mathbb{R}^3.$$

In particular, a *homogeneous* body has a plane of material symmetry if, and only if, it is symmetric with respect to that plane.

Proposition 6.1. *Assume that Σ has x_2x_3 as a plane of material symmetry. Then the resistance tensors have the form*

$$\mathbf{K} = \begin{pmatrix} K_{11} & 0 & 0 \\ 0 & K_{22} & K_{23} \\ 0 & K_{23} & K_{33} \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} B_{11} & 0 & 0 \\ 0 & B_{22} & B_{23} \\ 0 & B_{23} & B_{33} \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} 0 & C_{12} & C_{13} \\ C_{21} & 0 & 0 \\ C_{31} & 0 & 0 \end{pmatrix}.$$

Proof. Since Σ is invariant under the orthogonal transformation given by

$$\mathbf{Q} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \det \mathbf{Q} = -1,$$

then formulae (39) yield

$$K_{12} = K_{13} = B_{12} = B_{13} = 0, \\ C_{11} = C_{22} = C_{33} = C_{23} = C_{32} = 0. \quad \square$$

Now consider the system (36) which solves the problem of the steady free fall, in the case when the body Σ has a plane of material symmetry, say x_2x_3 . Suppose moreover that Σ is homogeneous, so that the center of mass and the centroid coincide, hence $\mathbf{r} = 0$. In such a case the second equation of (36) becomes

$$m_e(\mathbf{C}\mathbf{K}^{-1}\mathbf{C}^\top - \mathbf{B})^{-1}\mathbf{C}\mathbf{K}^{-1}\mathbf{g} = \lambda\mathbf{g}. \quad (40)$$

Since $\boldsymbol{\omega} = \lambda\mathbf{g}$, we get a translational solution whenever $\lambda = 0$. Being $(\mathbf{C}\mathbf{K}^{-1}\mathbf{C}^\top - \mathbf{B})$ and \mathbf{K} positive definite matrices, (40) has a solution $\lambda = 0$ if, and only if, $\det \mathbf{C} = 0$. In the case of a body with a plane of material symmetry, indeed, the latter condition is satisfied and it is easy to check that an eigenvector of \mathbf{C} , say \mathbf{u}_0 , corresponding to the eigenvalue $\lambda = 0$ lies in the plane x_2x_3 . Hence one has

$$m_e(\mathbf{C}\mathbf{K}^{-1}\mathbf{C}^\top - \mathbf{B})^{-1}\mathbf{C}\mathbf{K}^{-1}\mathbf{g} = 0 \iff \mathbf{g} = \mathbf{K}\mathbf{u}_0;$$

by the form of \mathbf{K} given in Proposition 6.1, also the vector $\mathbf{K}\mathbf{u}_0$ lies in the plane x_2x_3 . We can summarize the latter result in the following theorem:

Theorem 6.2. *Assume that Σ has x_2x_3 as a plane of material symmetry. Then there exist an orientation of the body, lying in the same plane of symmetry, which gives rise to a purely translational solution.*

Now it is quite easy to study the class of bodies with two orthogonal planes of symmetry:

Corollary 6.3. *If the body has two orthogonal planes of symmetry, say x_1x_3 and x_2x_3 , then the free fall along the x_3 -direction gives rise to a purely translational motion.*

Proof. Since the body is invariant under the orthogonal matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

it is easy to check that \mathbf{K} and \mathbf{B} are diagonal, and \mathbf{C} has the form

$$\mathbf{C} = \begin{pmatrix} 0 & C_{12} & 0 \\ C_{21} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence $\mathbf{u}_0 = (0, 0, a)$ is an eigenvector of \mathbf{C} corresponding to the null eigenvalue, and the motion with orientation given by

$$\mathbf{g} = \frac{\mathbf{K}\mathbf{u}_0}{|\mathbf{K}\mathbf{u}_0|} = (0, 0, \pm 1)$$

furnishes a purely translational solution. \square

6.2. Helicoidally symmetric bodies. Now we study bodies which are invariant under the action of a rotation of angle $\theta \in [0; 2\pi[$ around the x_1 -axis, which is represented by the orthogonal matrix

$$\mathbf{R}_\theta := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{pmatrix}.$$

Following [7], we say that a (one-dimensional) body Σ is *helicoidally symmetric* if there exists a co-moving frame such that

$$\mathbf{R}_\theta \Sigma = \Sigma \quad \text{for some } \theta \neq 0, \pi,$$

that is, if it is invariant under a discrete group of co-axial rotations of order strictly greater than 2. For instance, a homogeneous body composed by three concurrent edges of a regular tetrahedron is helicoidally symmetric with $\theta = 2\pi/3$.

Proposition 6.4. *Assume that Σ is helicoidally symmetric around x_1 . Then \mathbf{K} and \mathbf{B} are diagonal with $K_{22} = K_{33}$ and $B_{22} = B_{33}$, and \mathbf{C} is of the form*

$$\mathbf{C} = \begin{pmatrix} C_{11} & 0 & 0 \\ 0 & C_{22} & C_{23} \\ 0 & -C_{23} & C_{33} \end{pmatrix}.$$

Proof. Let us employ formulae (39) with $\mathbf{Q} = \mathbf{R}_\theta$, keeping into account that $\det \mathbf{R}_\theta = 1$. For the matrix \mathbf{C} we get the conditions

$$\begin{aligned} C_{12} &= C_{12} \cos \theta + C_{13} \sin \theta, \\ C_{13} &= C_{13} \cos \theta - C_{12} \sin \theta, \\ C_{21} &= C_{21} \cos \theta + C_{31} \sin \theta, \\ C_{22} &= C_{22} \cos^2 \theta + (C_{23} + C_{32}) \cos \theta \sin \theta + C_{33} \sin^2 \theta, \\ C_{23} &= C_{23} \cos^2 \theta + (C_{33} - C_{22}) \cos \theta \sin \theta - C_{32} \sin^2 \theta, \\ C_{31} &= C_{31} \cos \theta - C_{21} \sin \theta, \\ C_{32} &= C_{32} \cos^2 \theta + (C_{33} - C_{22}) \cos \theta \sin \theta - C_{23} \sin^2 \theta, \\ C_{33} &= C_{33} \cos^2 \theta - (C_{23} + C_{32}) \cos \theta \sin \theta + C_{22} \sin^2 \theta, \end{aligned}$$

which in turn imply that $C_{12} = C_{21} = C_{13} = C_{31} = 0$ and $C_{23} + C_{32} = 0$, since $\theta \neq 0, \pi$. Being \mathbf{K} and \mathbf{B} symmetric, we have the further conditions $K_{23} = B_{23} = 0$ and $K_{22} = K_{33}$, $B_{22} = B_{33}$. \square

6.3. Helicoidally symmetric bodies with fore-aft symmetry. A remarkable situation is the case of a homogeneous one-dimensional helicoidally symmetric body with *fore-aft symmetry*, that is, a body which is both helicoidally symmetric around an axis, and has a plane of symmetry orthogonal to that axis.

Without loss of generality, let us assume that a one-dimensional body Σ is helicoidally symmetric around x_1 and has x_2x_3 as a plane of symmetry. Since the coupling tensor \mathbf{C} has to satisfy both Proposition 6.1 and Proposition 6.4 at the same time, it follows that

$$\mathbf{C} = \mathbf{0}.$$

Assuming that Σ be homogeneous, so that $\mathbf{r} = \mathbf{0}$, the system (36) merely becomes

$$\begin{cases} \boldsymbol{\xi} = m_e \mathbf{K}^{-1} \mathbf{g}, \\ \mathbf{0} = \lambda \mathbf{g}, \end{cases}$$

hence $\lambda = 0$ for any direction \mathbf{g} . Then $\boldsymbol{\omega} = \mathbf{0}$ and for any given orientation the body falls with a purely translational velocity given by $\boldsymbol{\xi} = m_e \mathbf{K}^{-1} \mathbf{g}$.

APPENDIX A. GREEN'S FUNCTION FOR STOKES FLOW

The basic tool used to construct solutions to the Stokes problem is the so-called Stokeslet, that is the Green's function for the Stokes operator in \mathbb{R}^3 . In this Appendix⁽²⁾ we want to compute the expression of the Stokeslet in the case of our hyperviscous fluid, identified by the operator

$$\mathcal{A} := \ell^2 \Delta \Delta - \Delta.$$

We first need a Green's function g solution of the fourth-order elliptic equation

$$\ell^2 \Delta \Delta g - \Delta g = \delta(\mathbf{x} - \mathbf{x}').$$

Using the Fourier transform, we easily obtain

$$g(\mathbf{x} - \mathbf{x}') = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{e^{i\mathbf{k} \cdot (\mathbf{x} - \mathbf{x}')}}{|\mathbf{k}|^2 (\ell^2 |\mathbf{k}|^2 + 1)} d\mathbf{k}.$$

We choose a basis for the momentum space in such a way that $\mathbf{x} - \mathbf{x}'$ is along the k_3 -direction, set $R = |\mathbf{x} - \mathbf{x}'|$, switch to polar coordinates (k, θ, ϕ) , and use the calculus of residues to obtain

$$\begin{aligned} g(\mathbf{x} - \mathbf{x}') &= \frac{2\pi}{(2\pi)^3 \ell^2} \int_0^{+\infty} \int_{-1}^1 \frac{e^{ikR \cos \theta}}{k^2 + 1/\ell^2} d(\cos \theta) dk \\ &= \frac{2}{(2\pi)^2 \ell^2 R} \int_0^{+\infty} \frac{\sin kR}{k(k^2 + 1/\ell^2)} dk \\ &= \frac{1}{(2\pi)^2 \ell^2 R} \operatorname{Im} \left[\int_{-\infty}^{+\infty} \frac{e^{ikR}}{k(k^2 + 1/\ell^2)} dk \right] \\ &= \frac{1}{(2\pi)^2 \ell^2 R} \left(\pi \ell^2 - \pi \ell^2 e^{-\frac{R}{\ell}} \right). \end{aligned}$$

Hence, the Green's function is

$$g(\mathbf{x} - \mathbf{x}') = \frac{1}{4\pi |\mathbf{x} - \mathbf{x}'|} \left[1 - \exp \left(-\frac{|\mathbf{x} - \mathbf{x}'|}{\ell} \right) \right]. \quad (41)$$

Notice that, in the limit $\ell \rightarrow 0$, (41) reduces to the fundamental solution

$$g_1(\mathbf{x} - \mathbf{x}') = \frac{1}{4\pi |\mathbf{x} - \mathbf{x}'|} \quad (42)$$

for the Laplace operator. Moreover, g is well defined for any $\mathbf{x} \in \mathbb{R}^3$, at variance with the classical expression g_1 , which is singular at the origin.

We now proceed to construct the *hyperviscous Stokeslet*, that is a pressure field $p\boldsymbol{\zeta}$ and a velocity field $\boldsymbol{\zeta}$ satisfying

$$\operatorname{div} \boldsymbol{\zeta} = 0, \quad (43)$$

⁽²⁾The results of the Appendix are based on [5].

$$\nabla p_\zeta + \mathcal{A}\zeta = \mathbf{h}\delta(\mathbf{x}), \quad (44)$$

with $\mathbf{h} \in \mathbb{R}^3$ and $\mathcal{A} = \ell^2 \Delta \Delta - \Delta$. Let ϕ satisfy $\mathcal{A}\phi = \delta(\mathbf{x} - \mathbf{x}')$; then, since \mathcal{A} commutes with ∇ , a solution for (43)–(44) is given by

$$p_\zeta = -\mathcal{A}\vartheta, \quad \zeta = \mathbf{h}\phi + \nabla\vartheta.$$

The scalar field ϑ entering this solution is chosen to satisfy the constraint (43) and turns out to have the explicit form

$$\vartheta = (-\Delta)^{-1}(\mathbf{h} \cdot \nabla\phi) = g_1 * (\mathbf{h} \cdot \nabla\phi),$$

where g_1 is as defined in (42) and $*$ denotes the usual convolution product. Now, exploiting the properties of the convolution and the operator \mathcal{A} , and using the Green's function g given by equation (41), we find that

$$-(g_1 * \mathcal{A}(\mathbf{h} \cdot \nabla g)) = -(g_1 * \operatorname{div}(\mathcal{A}(g\mathbf{h}))) = -\operatorname{div}(g_1 * \mathcal{A}(g\mathbf{h})) = -\mathbf{h} \cdot \nabla g_1$$

and, denoting by \widehat{f} the Fourier transform of the function f ,

$$\begin{aligned} g_1 * (\mathbf{h} \cdot \nabla g) &= \frac{1}{(2\pi)^3} \int i(\mathbf{h} \cdot \mathbf{k}) \widehat{g_1} \widehat{g} e^{i\mathbf{k} \cdot \mathbf{x}} d\mathbf{k} \\ &= \frac{1}{(2\pi)^3} \int \frac{i(\mathbf{h} \cdot \mathbf{k}) e^{i\mathbf{k} \cdot \mathbf{x}}}{|\mathbf{k}|^4 (\ell^2 |\mathbf{k}|^2 + 1)} d\mathbf{k} \\ &= \frac{\mathbf{h} \cdot \mathbf{x}}{4\pi^2 \ell^2 |\mathbf{x}|} \int_0^{+\infty} \int_{-1}^1 \frac{i \cos \theta e^{ik|\mathbf{x}| \cos \theta}}{k(k^2 + 1/\ell^2)} d(\cos \theta) dk \\ &= \frac{-\mathbf{h} \cdot \mathbf{x}}{4\pi^2 \ell^2 |\mathbf{x}|} \int_{-1}^1 \tau \int_0^{+\infty} \frac{\sin(k|\mathbf{x}| \tau)}{k(k^2 + 1/\ell^2)} dk d\tau \\ &= \frac{-\mathbf{h} \cdot \mathbf{x}}{8\pi |\mathbf{x}|} \int_{-1}^1 |\tau| \left(1 - e^{-\frac{|\mathbf{x}|}{\ell} |\tau|}\right) d\tau \\ &= -\frac{\mathbf{h} \cdot \mathbf{x}}{8\pi |\mathbf{x}|} \left[1 + \frac{2\ell}{|\mathbf{x}|} e^{-\frac{|\mathbf{x}|}{\ell}} + \frac{2\ell^2}{|\mathbf{x}|^2} \left(e^{-\frac{|\mathbf{x}|}{\ell}} - 1\right)\right]. \end{aligned}$$

Hence the Stokeslet is given by

$$p_\zeta(\mathbf{x}) = \frac{\mathbf{h} \cdot \mathbf{x}}{4\pi |\mathbf{x}|^3},$$

$$\begin{aligned} \zeta(\mathbf{x}) &= \frac{\mathbf{h}}{8\pi |\mathbf{x}|} \left[1 - 2e^{-\frac{|\mathbf{x}|}{\ell}} - \frac{2\ell}{|\mathbf{x}|} e^{-\frac{|\mathbf{x}|}{\ell}} - \frac{2\ell^2}{|\mathbf{x}|^2} \left(e^{-\frac{|\mathbf{x}|}{\ell}} - 1\right)\right] \\ &\quad + \frac{(\mathbf{h} \cdot \mathbf{x}) \mathbf{x}}{8\pi |\mathbf{x}|^3} \left[1 + 2e^{-\frac{|\mathbf{x}|}{\ell}} + \frac{6\ell}{|\mathbf{x}|} e^{-\frac{|\mathbf{x}|}{\ell}} + \frac{6\ell^2}{|\mathbf{x}|^2} \left(e^{-\frac{|\mathbf{x}|}{\ell}} - 1\right)\right]. \end{aligned}$$

We also define the hyperviscous Oseen tensor \mathbf{Z} as, using Cartesian components,

$$\begin{aligned} Z_{ij}(\mathbf{x}) &:= \frac{\delta_{ij}}{8\pi |\mathbf{x}|} \left[1 - 2e^{-\frac{|\mathbf{x}|}{\ell}} - \frac{2\ell}{|\mathbf{x}|} e^{-\frac{|\mathbf{x}|}{\ell}} - \frac{2\ell^2}{|\mathbf{x}|^2} \left(e^{-\frac{|\mathbf{x}|}{\ell}} - 1\right)\right] \\ &\quad + \frac{x_i x_j}{8\pi |\mathbf{x}|^3} \left[1 + 2e^{-\frac{|\mathbf{x}|}{\ell}} + \frac{6\ell}{|\mathbf{x}|} e^{-\frac{|\mathbf{x}|}{\ell}} + \frac{6\ell^2}{|\mathbf{x}|^2} \left(e^{-\frac{|\mathbf{x}|}{\ell}} - 1\right)\right], \end{aligned}$$

whereby it follows that $\zeta(\mathbf{x}) = \mathbf{Z}(\mathbf{x})\mathbf{h}$. The Stokeslet allows us to obtain an integral representation for the solution of

$$\nabla p - \Delta(\mathbf{u} - \ell^2 \Delta \mathbf{u}) = \rho \mathbf{b},$$

with vanishing condition at infinity, in the form of a convolution:

$$\mathbf{u}(\mathbf{x}) := \rho \int_{\mathbb{R}^3} \mathbf{Z}(\mathbf{x} - \mathbf{x}') \mathbf{b}(\mathbf{x}') d\mathbf{x}'. \quad (45)$$

In particular, whenever \mathbf{b} has compact support, such as in the case of the gravity force acting on a bounded body Σ , the solution \mathbf{u} behaves as $1/|\mathbf{x}|$ for $|\mathbf{x}| \rightarrow \infty$.

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